ANALYSIS 1

(ENGLISH)

LECTURE 14

DEF $f : E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$

TAKE $x_0 \in \mathbb{R}$. ASSUME E CONTAINS A POINTED NEIGHBORHOOD OF x_0.

THEN $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$ IF

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ S.T.

IF $\underline{\underline{0 <}} |x - x_0| \leq \delta_\varepsilon \Longrightarrow |f(x) - \ell| \leq \varepsilon$

THM $f: E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME E CONTAINS A POINTED NEIGHB. OF x_0. TFAE

1. $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$

2. FOR ALL $\boxed{(x_n) \subseteq E \setminus \underline{\underline{\{x_0\}}}}$ *

IF $x_n \xrightarrow{n \to \infty} x_0 \implies f(x_n) \xrightarrow{n \to \infty} \ell$

SEQUENCE

SINCE $x_n \in D(f)$

EXAMPLE

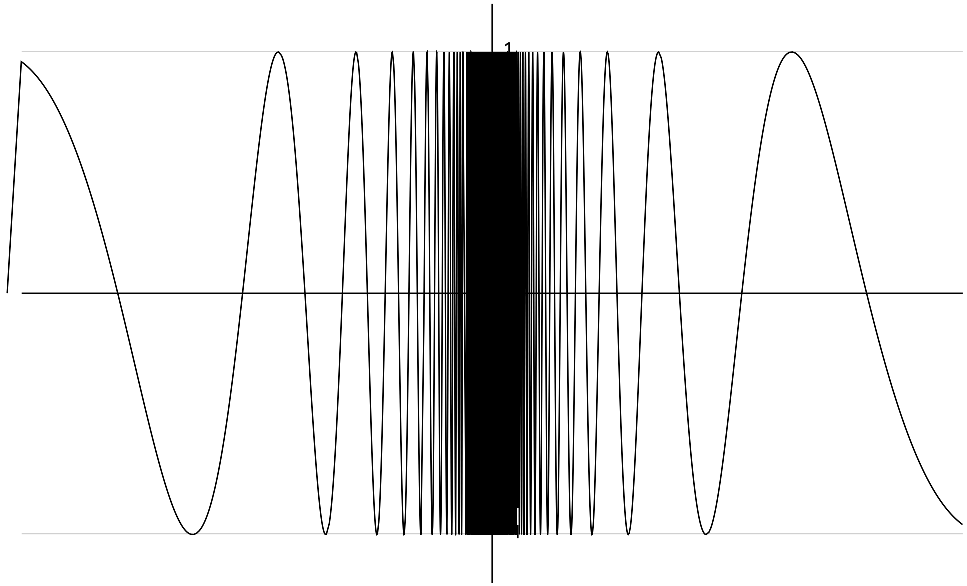
$f(x) = x^2 + 1 \qquad f : \mathbb{R} \to \mathbb{R}$

$$\lim_{x \to 5} (x^2 + 1) = \ell$$

$$\Big\Updownarrow \text{ THM}$$

$$\lim_{n \to \infty} (\underbrace{x_n^2 + 1}_{= f(x_n)}) = \ell \qquad \left[\forall (x_n) \subseteq \mathbb{R} \setminus \{5\},\ x_n \xrightarrow{n\to\infty} 5 \right]$$

$$= \left[\lim_{n \to \infty} x_n^2 \right] + 1 = \left(\lim x_n \right)^2 + 1 = 26$$



$$f : \mathbb{R}^* \to \mathbb{R}$$

$$f(x) = \sin\left(\frac{1}{x}\right)$$

$$\lim_{x \to 0} f(x) = \ ? \quad \text{[POLL]}$$

$$\lim f(x_n) =$$

$$x_n = \frac{1}{\pi/2 + 2n\pi}$$

$$y_n = \frac{1}{-\pi/2 - 2n\pi}$$

$$x_n, y_n \xrightarrow{n \to \infty} 0$$

1 $= 1$

2 $= 0$

3 $= \pi$

~~4~~ NONE OF THE ABOVE lim DOES NOT EXIST.

$$f(x_n) = \sin\left(\pi/2 + 2n\pi\right) = 1$$

$$f(y_n) = \sin\left(-\frac{\pi}{2}\right) = -1$$

$$\lim f(y_n)$$

THM

$f, g : E \longrightarrow \mathbb{R}$, $x_0 \in \mathbb{R}$

Assume $\lim_{x \to x_0} f(x) = F \in \mathbb{R}$, $\lim_{x \to x_0} g(x) = G \in \mathbb{R}$

THEN

1. $\lim_{x \to x_0} f(x) + g(x) = F + G$
2. $\lim_{x \to x_0} f(x) \cdot g(x) = F \cdot G$
3. IF $G \neq 0$, $g(x) \neq 0 \ \forall x \in E$

   $\lim_{x \to x_0} f(x)/g(x) = F/G$

SAME AS THE OLD RULES FOR SEQ'S

$$\lim_{x \to x_0} f(x) = F \quad , \quad \lim_{x \to x_0} g(x) = G$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\forall (y_n) \subseteq E \setminus \{x_0\}$$

$$y_n \xrightarrow{n \to \infty} x_0$$

THM $f : E \longrightarrow \mathbb{R}$, $E \subseteq \mathbb{R}$, $x_0 \in \mathbb{R}$

ASSUME E CONTAINS A POINTED NEIGHB. OF x_0. TFAE

1. $\lim_{x \to x_0} f(x) = \ell \in \mathbb{R}$

2. FOR ALL $\boxed{(x_n) \subseteq E \setminus \{x_0\}}$

IF $x_n \xrightarrow{n \to \infty} x_0 \Longrightarrow f(x_n) \xrightarrow{n \to \infty} \ell$

SEQUENCE

$$\lim_{x\to x_0} f(x) = F \quad , \quad \lim_{x\to x_0} g(x) = G$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\forall (y_n) \subseteq E \setminus \{x_0\}$$

$$y_n \xrightarrow{n\to\infty} x_0$$

$$\Leftarrow \qquad\qquad \Rightarrow$$

$$f(y_n) \xrightarrow{n\to\infty} F \qquad\qquad g(y_n) \xrightarrow{n\to\infty} G$$

$$\Downarrow$$

CONV

$$\forall (y_n) \subseteq E \setminus \{x_0\},\ y_n \xrightarrow{n\to\infty} x_0 \Rightarrow \lim_{n\to\infty} \overline{f(y_n)} + \overline{g(y_n)} = F + G$$

$$\lim f(y_n) + \lim g(y_n)$$

SQUEEZE THM FOR LIMITS

THM $f, g, h : E \to \mathbb{R}$ $x_0 \in \mathbb{R}$

ASSUME THAT ON A PUNCTURED NEIGHB. S OF x_0 $\subseteq E$ "POINTED"

$]x_0-\delta, x_0+\delta[\setminus \{x_0\}$, $\delta>0$

$$f(x) \le g(x) \le h(x) \quad \forall x \in S$$

IF $\lim_{x\to x_0} f(x) = L$
$\lim_{x\to x_0} h(x) = L$ $\Longrightarrow$ $\lim_{x\to x_0} g(x) = L$

EXAMPLE

$$\lim_{x\to 0} \cos(x) = 1 \iff \lim_{x\to 0} \cos(x) - 1 = 0$$

$$\Big\Vert$$

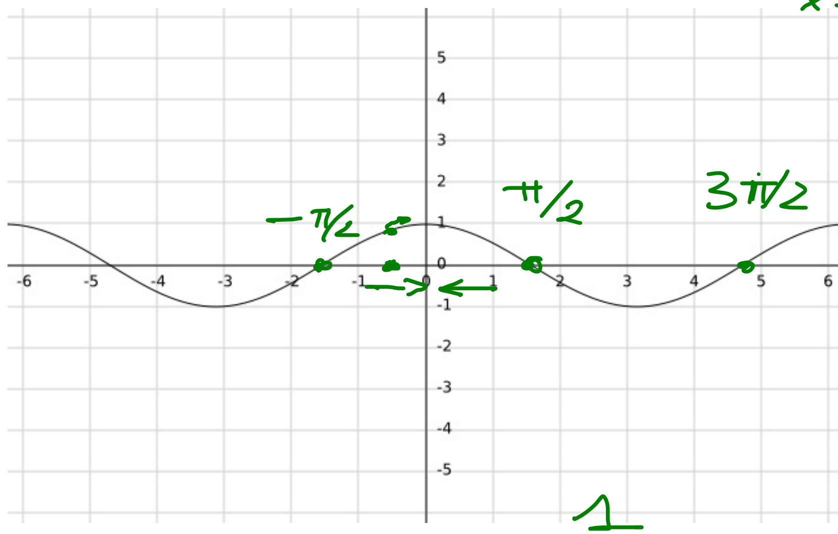
$$\lim_{x\to 0} \cos(x) - \lim_{x\to 0} 1 = 0$$

$$\lim_{x\to 0} 1 = 1$$

$$\Updownarrow$$

$$\lim_{x\to 0} |\cos(x)-1| = 0$$

$$[\cos(0) = 1]$$



$$0 \le |\cos x - 1| \le \underbrace{|\cos(x)+1|}_{\ge 1}\,|\cos(x)-1| =$$

HOLDS for $x \in [-\pi/2, \pi/2]$

$$|1-\cos^2(x)| = |\sin^2(x)| \le |x|^2$$

TO BE SEEN SOON

$$0 \le |\cos x - 1| \le |x|^2$$

$$\downarrow x \to 0$$

$$0$$

$$\forall x \in [-\pi/2, \pi/2]$$

$$\lim_{x \to 0} 0 = 0 = \lim_{x \to 0} |x|^2$$

$$|x| = \begin{cases} x & \text{for } x \ge 0 \\ -x & \text{for } x < 0 \end{cases}$$

$$\lim_{x \to 0} |x| = 0 \iff \forall (x_n) \text{ s.t. }$$

$$x_n \to 0 \Rightarrow |x_n| \to 0$$

EXAMPLE

$$\lim_{x \to 0} \cos(x) = 1 \ ?$$

$$\Updownarrow$$

$$\lim_{x \to 0} \cos(x) - 1 = 0$$

$$0 \leqslant \left| \cos(x) - 1 \right| \leqslant$$

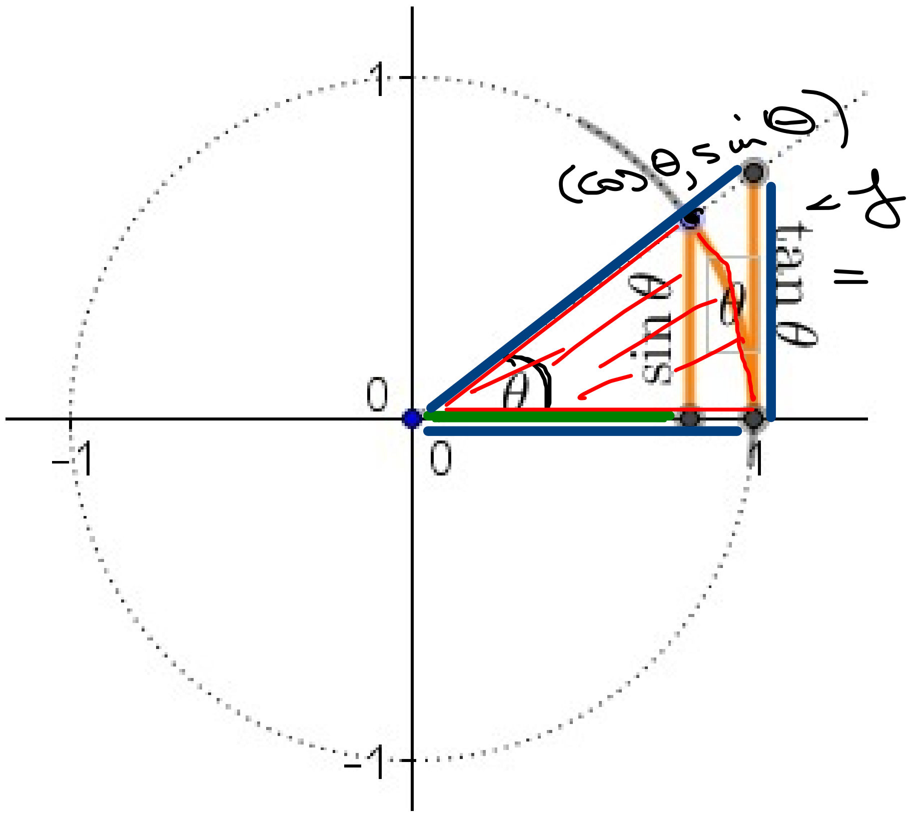
$$\lim_{x \to 1} \frac{\sin(x-1)}{(x-1)}$$

$$t \doteq x - 1 \qquad \Updownarrow$$

$$\lim_{t \to 0} \frac{\sin(t)}{t} = ?$$

TO COMPUTE
W/ SQUEEZE
THM

$$\frac{1}{\cos(\theta)} = \frac{y}{\sin\theta} \Rightarrow y = \tan(\theta)$$



$$= \frac{\sin\theta}{\cos\theta}$$

$$\text{Area } \Delta = \frac{\tan\theta}{2}$$

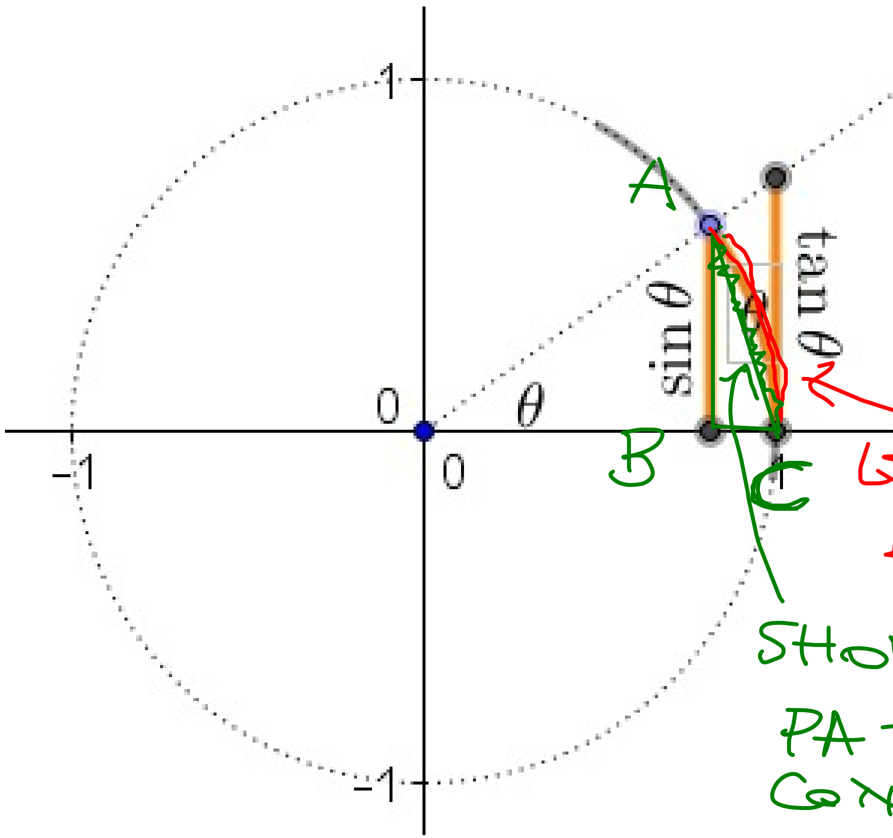
$$\text{Area}(C(\theta)) = \frac{\theta}{2}$$

For $-\pi/2 \leq \theta \leq \pi/2$

$$\left|\frac{\theta}{\cancel{2}}\right| \leq \left|\frac{\tan\theta}{\cancel{2}}\right|$$

$$\cos\theta \leq \frac{\sin\theta}{\theta}$$

$$\theta \in [-\pi/2, \pi/2] \setminus \{0\}$$



$$|\sin(\theta)| \le |\theta|$$

$$\overline{AB}^2 + \overline{BC}^2 = \overline{CA}^2$$

$$\overline{CA} \ge \overline{AB} = |\sin\theta|$$

LENGTH = θ

SHORTEST PATH CONNECTING A & C

CIRCUMF. = 2π

$$|\sin\theta| \le \overline{CA} \le |\theta| \quad \text{for } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$|\sin(\theta)|/|\theta| \le 1$$

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

$$\forall x \in \left[-\pi/2 , \pi/2\right]$$

$$\lim_{x \to 0} \cos(x) = 1 = \lim_{x \to 0} 1$$

$$\Downarrow$$

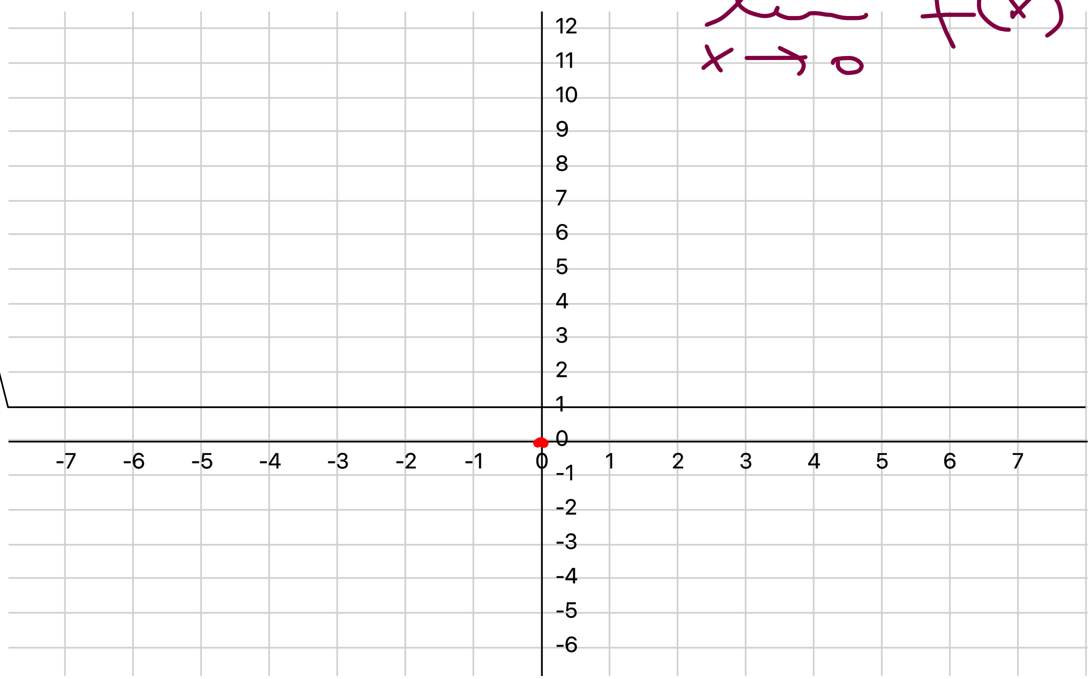
$$\frac{\sin x}{x} \xrightarrow{x \to 0} 1$$

EXAMPLE

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \quad (*) \\ 0 & \text{if } x = 0 \end{cases}$$

$\neq \lim_{x \to 0} f(x)$



$$\lim_{x \to 0} f(x) = 1$$

$$\Updownarrow$$

$\forall (x_n) \subseteq \mathbb{R} \setminus \{0\}$.

S.T. $x_n \to 0$

THEN $f(x_n) \to 1$

SINCE $(x_n) \subseteq \mathbb{R} \setminus \{0\} \Rightarrow \forall n \; f(x_n) \overset{(*)}{=} 1$

LIMITS

EXAMPLE

$$f : E \longrightarrow \mathbb{R}$$

$$f(x) = \frac{\sin(x-1)}{(x-1)}$$

$$E = \mathbb{R} \setminus \{1\}$$

Q: CAN WE EXTEND $\tilde{f} : \mathbb{R} \to \mathbb{R}$?

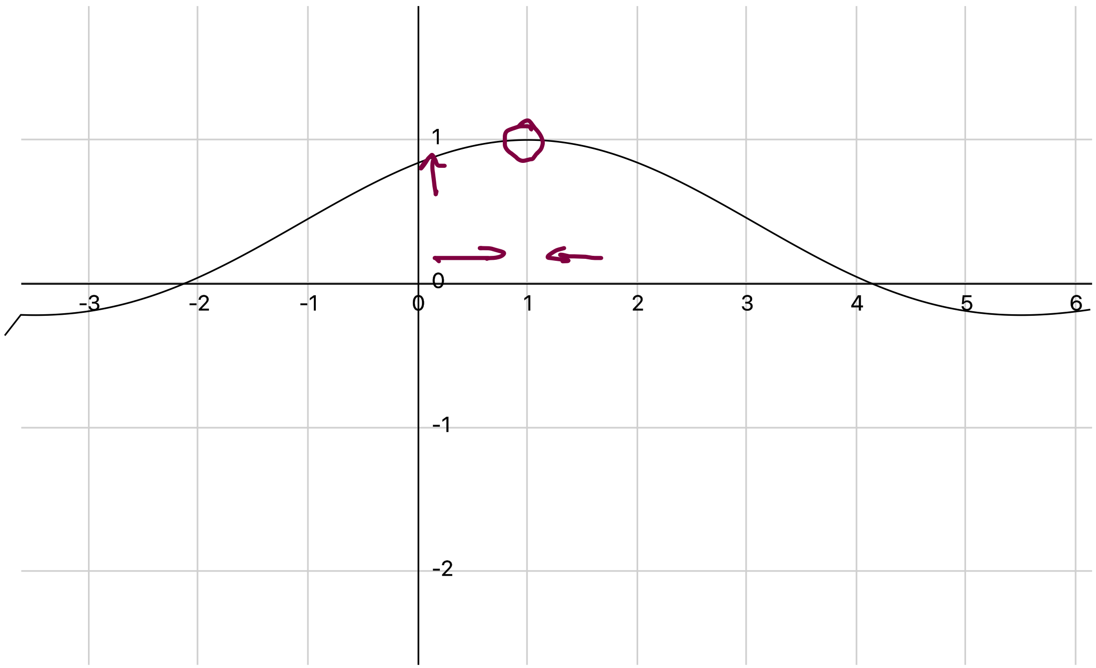
$$\tilde{f}(x) = \begin{cases} \dfrac{\sin(x-1)}{(x-1)} & \forall x \neq 1 \\ 1 & x = 1 \end{cases}$$

LIMITS

EXAMPLE

$$\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$$

$$\tilde{f}(x) \doteq f(x) = \frac{\sin(x-1)}{(x-1)}, \quad x \neq 1 \qquad \tilde{f}(1) = 1$$



$$\lim_{x \to 1} f(x) = 1$$

$$\parallel$$

$$\lim_{x \to 1} \frac{\sin(x-1)}{x-1}$$

$$\parallel$$

$$\lim_{t \to 0} \frac{\sin(t)}{t} = 1$$

DEF $f : E \to \mathbb{R}$, $x_0 \in E = D(f)$

$\cup|$

f IS CONTINUOUS AT x_0 $(x_0 - \delta, x_0 + \delta)$, $\delta > 0$

IF $\lim\limits_{x \to x_0} f(x) = f(x_0)$

DEF $f : E \to \mathbb{R}$, $x_0 \in E$

① f IS CONTINUOUS AT x_0

IF $\lim_{x \to x_0} f(x) = f(x_0)$

② f is CONTINUOUS IF

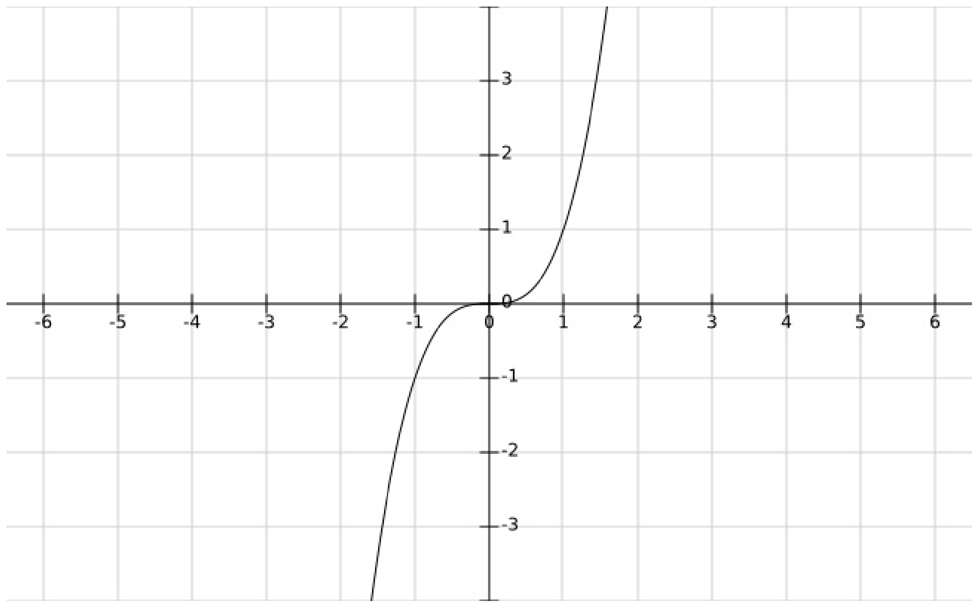
IT IS CONTINUOUS AT x_0, $\forall x_0 \in E$

NOTATION: $f \in C^0(E)$

EXAMPLE

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = x^3$$



$$\lim_{x \to 1} f(x) = 1 = f(1)$$

f IS CONTINUOUS AT $1 \in D(f) = \mathbb{R}$

IN GENERAL

$f : \mathbb{R} \longrightarrow \mathbb{R}$ POLYNOMIAL

$$f(x) = \sum_{i=0}^{k} a_i x^i$$

$a_i \in \mathbb{R}$ CONST. COEFF'S OF THE POLYNOMIAL

f is CONTINUOUS AT ANY POINT $x_0 \in \mathbb{R}$

$$\left[f \in C^0(\mathbb{R}) \right]$$

IN GENERAL

$f : \mathbb{R} \to \mathbb{R}$ POLYNOMIAL

$$f(x) = \sum_{i=0}^{K} a_i x^i$$

f is CONTINUOUS AT ANY POINT $x_0 \in \mathbb{R}$

$$\Updownarrow$$

$$\lim_{x \to x_0} f(x) = f(x_0) \quad [\forall x_0 \in \mathbb{R}]$$

$\mathbb{R} \setminus \{x_0\}$

$$\Updownarrow$$

$$\forall x_n \xrightarrow{n\to\infty} x_0 , \quad \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sum_{i=0}^{K} \underline{a_i x_n^i} =$$

$$= \sum_{i=0}^{K} a_i \left(\lim_{n\to\infty} x_n\right)^i = \sum_{i=0}^{K} a_i x_0^i = f(x_0)$$

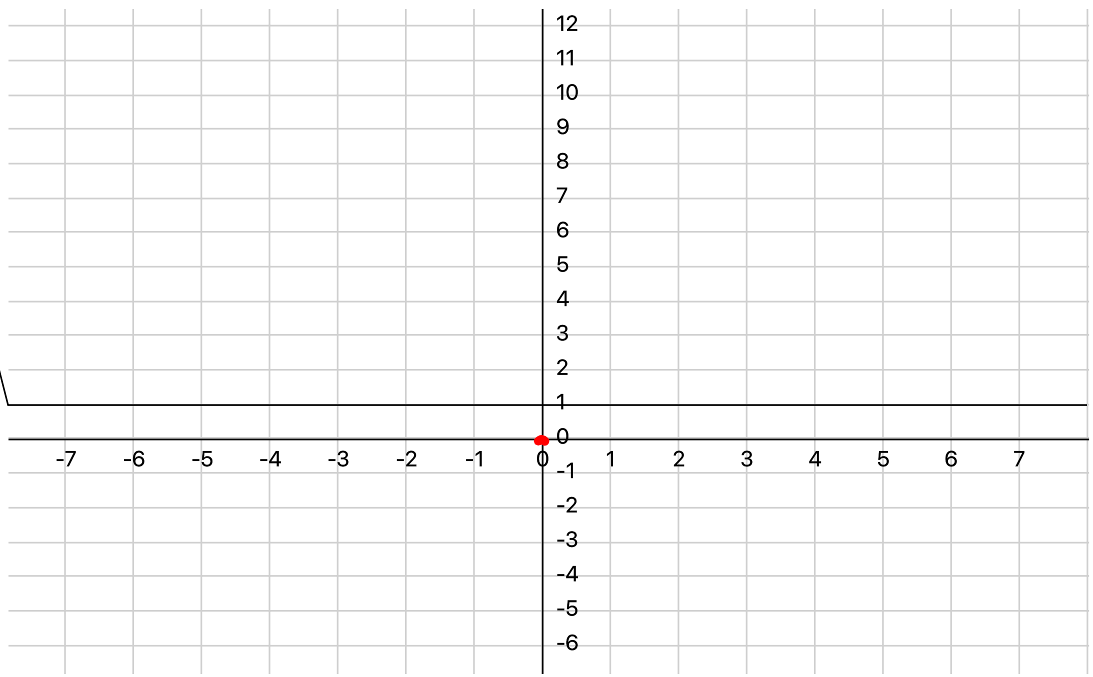
$\hookrightarrow$ conv.

$a_i x_n^i \to a_i x_0^i$

EXAMPLE

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$



$f(0) = 0$

$\lim_{x \to 0} f(x) = 1 \neq f(0)$

f is NOT CONTIN.
AT $x = 0$

DEF $f: E \to \mathbb{R}$, $\underline{\underline{x_0 \in E}}$

f IS CONTINUOUS AT x_0

IF $\displaystyle\lim_{x \to x_0} f(x) = \underline{\underline{f(x_0)}}$

$\Updownarrow$ DEF. OF LIMIT.

$\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ S.T. $\underline{|x - x_0| \leq \delta_\varepsilon} \Rightarrow |f(x) - f(x_0)| \leq \varepsilon$

$\Updownarrow$

$\forall \boxed{(x_n) \subseteq E}$ IF $x_n \xrightarrow{n\to\infty} x_0 \Longrightarrow f(x_n) \to f(x_0)$

$\cancel{E \setminus \{x_0\}}$

EXAMPLE

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

TAKE $\bar{x} \in \mathbb{Q}$

$$\lim_{x \to \bar{x}} f(x) = ?$$

f IS NOT CONT. AT ANY $x_0 \in \mathbb{R}$

$\Rightarrow$ f is NOT C^0 AT $\bar{x}$

CLAIM: THE LIMIT DOES NOT EXIST.

$x_n = \bar{x} + \frac{1}{n} \longrightarrow \bar{x}$ $\quad \forall n \geq 1$ $\quad x_n \in \mathbb{Q}$

$y_n = \bar{x} + \frac{\sqrt{2}}{n} \nearrow$ $\quad y_n \in \mathbb{R} \setminus \mathbb{Q}$

$f(x_n) = 1$ $\quad \forall n \geq 1$

$f(y_n) = 0$

EXAMPLE

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

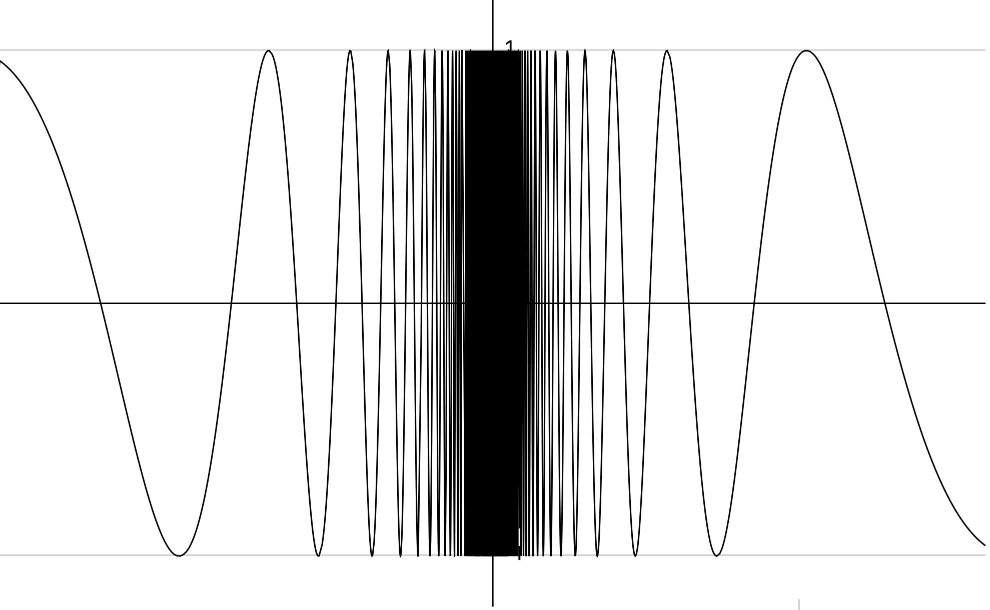
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

TAKE $\bar{x} \in \mathbb{Q}$

$$\lim_{x \to x_0} f(x) = ?$$

$$x_k = \bar{x} + \frac{1}{n}$$

$$y_k = \bar{x} - \frac{1}{\sqrt{2}\, n}$$

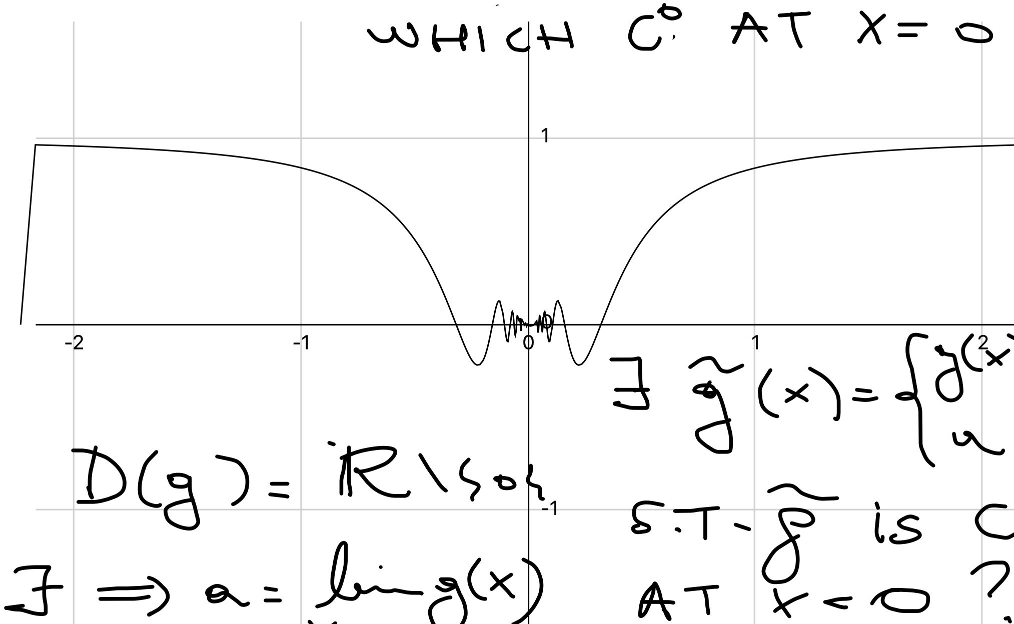


$$f(x) = \sin\left(\frac{1}{x}\right)$$

$\nexists$ ANY EXT.

$$\tilde{f}(x) = \begin{cases} f(x) & x \neq 0 \\ a & x = 0 \end{cases}$$

WHICH C^0 AT $x = 0$



$$g(x) = x \cdot \sin\left(\frac{1}{x}\right)$$

CONTINUOUS AT 0?

$D(g) = \mathbb{R} \setminus \{0\}$

$$\exists\ \tilde{g}(x) = \begin{cases} g(x) & x \neq 0 \\ a & x = 0 \end{cases}$$

S.T. $\tilde{g}$ is C^0 AT $x = 0$?

IF $\tilde{g}$ $\exists \Rightarrow a = \lim_{x \to 0} g(x)$

$a = 0$:)

$$0 \leq \left| g(x) = x \sin\left(\frac{1}{x}\right) \right| \leq |x|$$

$$\lim_{x \to 0} |x| = 0 \overset{\text{SQ.}}{\Longrightarrow} \lim_{x \to 0} |g(x)| = 0$$

$$\Updownarrow$$

$$x \to 0 \qquad \lim_{x \to 0} g(x) = 0$$

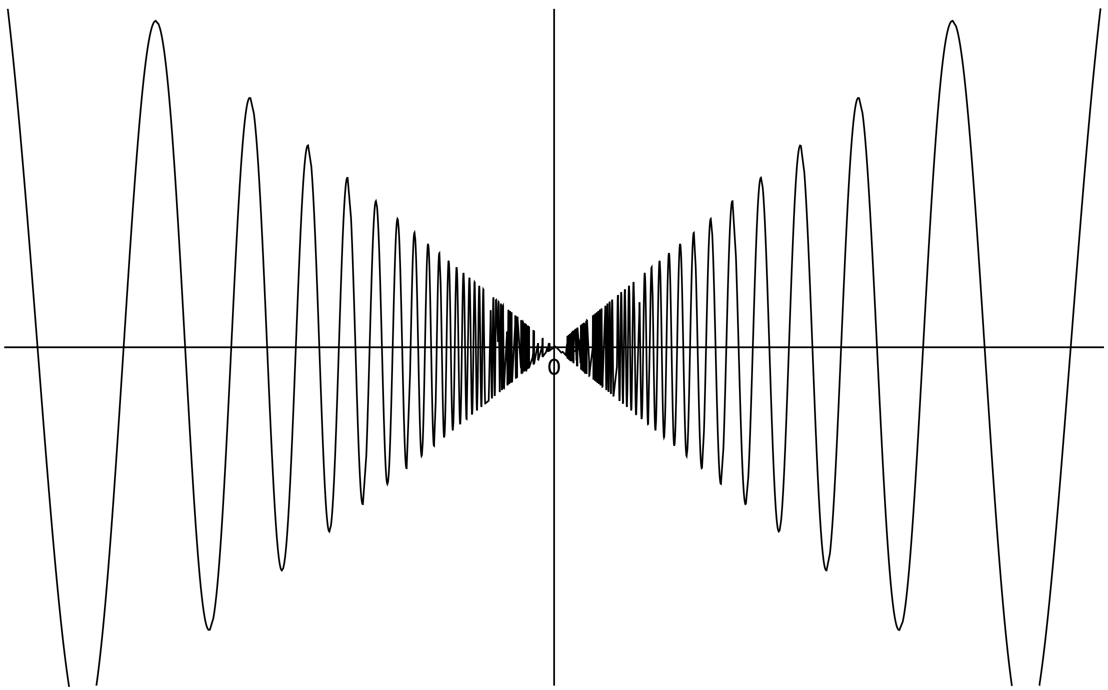
$$\lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \overset{t=\frac{1}{x}}{\text{“}=\text{”}} \lim_{\substack{t \to +\infty \\ t \to -\infty}} \frac{\sin t}{t}$$

$$g(x) = x \cdot \sin\left(\frac{1}{x}\right)$$

CONTINUOUS AT 0?

$$g : \mathbb{R}^* \to \mathbb{R}$$

$$\lim_{x \to 0} g(x) = ? \quad \text{[POLL]}$$



1. $= 1$
2. $= 0$
3. g UNBOUNDED AROUND 0
4. g BOUNDED BUT NO lim

PROP $f, g : E \longrightarrow \mathbb{R}$, $\underline{\underline{x_0 \in E}}$

ASSUME f, g ARE CONTINUOUS AT x_0

THEN

$[(\alpha f + \beta g)(x) \doteq \alpha \cdot f(x) + \beta \cdot g(x)]$ "CONTINUOUS"

(1) $\forall \alpha, \beta \in \mathbb{R}$ $\alpha f + \beta g$ IS C^0 AT x_0

(2) $f \cdot g$ IS C^0 AT x_0 $[(f \cdot g)(x) \doteq f(x) \cdot g(x)]$

(3) IF $g(x) \neq 0$ $\forall x \in E$ $\Longrightarrow$ $\dfrac{f}{g}$ IS C^0 AT x_0

$\left(\forall x \in (x_0 - \delta, x_0 + \delta) \right)$
$\delta > 0$